Complementary Majority Domination Number

S. Anandha Prabhavathy

Abstract - A function f : V → {−1, +1} is called a Complementary Majority Dominating Function namely (CMDF) of G if $\sum_{u \notin N[v]} f(u) \geq 1$ for at least half of the vertices v ∈ V (G) with deg v ≠ n − 1. The Complementary Majority Domination Number of G is denoted by $\gamma_{Cmaj}(G)$ and is defined as $\gamma_{Cmaj}(G)$ = min {w(f) | f is a minimal CMDF of G}. In this paper, we initiate the study of complementary majority domination number in graphs.

Keywords: Complementary Majority Dominating Function, Complementary Majority Domination Number.

1 Introduction

By a graph G = (V, E), we mean a finite, non-trivial, connected, and undirected graph with neither loops nor multiple edges. The order and size of G are denoted by n and m, respectively. For graph theoretic terminology we refer to Chartand and Lesniak [1].

The study of domination is one of the fastest growing areas within graph theory. A subset D of vertices is said to be a dominating set of G if every vertex in V either belongs to D or is adjacent to a vertex in D. The domination number γ(G) is the minimum cardinality of a dominating set of G. Survey of several advanced topics on domination is given in the book edited by Haynes et al. [2].

For a real valued function f : V → R on V , weight of f is defined to be w(f) = $\sum_{v \in V} f(v)$ and also for a subset S ⊆ V , we define f (S) = $\sum_{v \in S} f(v)$.Therefore, w(f) = f (V). Further, for a vertex v ∈ V , let f [v] = f (N [v]) for notation convenience. A function f : V → {−1, +1} is called a majority dominating function if f [v] ≥ 1 for at least half of the vertices in G. The majority domination number of G is denoted by $\gamma_{maj}(G)$ and is defined as $\gamma_{maj}(G)$= min {w(f) | f is a majority dominating function of G}. Majority domination was first introduced by Broere et al. in [5] and further studied in [3].

A function f : V → {−1, +1} is called a Complementary Signed Dominating Function of G if $\sum_{u \notin N[v]} f(u) \geq 1$ for every v ∈ V (G) with deg v ≠ n − 1. The Complementary Signed Domination Number of G is defined as $\gamma_{cs}(G)$ =min {w(f) | f is a minimal complementary signed dominating function of G}. The parameter $\gamma_{cs}(G)$ was first investigated in [6].

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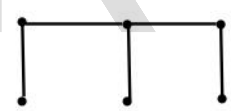
2 $\gamma_{cmaj}(G)$ for some special classes for graphs

Definition 2.1

A function $f: V \to \{-1, +1\}$ is called a Complementary Majority Dominating Function (CMDF) of G if $\sum_{u \notin N[v]} f(u) \geq 1$ for at least half of the vertices $v \in V(G)$ with $\deg v \neq n - 1$. The Complementary Majority Domination Number of G is denoted by $\gamma_{Cmaj}(G)$ and is defined as $\gamma_{Cmaj}(G) = \min\{w(f) \mid f \text{ is a minimal CMDF of } G\}$.

Example 2.2

Now consider the graph G as follows



By assigning +1 to the pendant vertices and -1 to the remaining vertices, we obtain $\sum_{u \notin N[v]} f(u) \geq 1$ for the vertices of degree greater than one. Hence $\gamma_{Cmaj}(G) = 0$.

Remark 2.3

If f is a complementary majority dominating function of a graph G, we define the sets P_f,M_f, and N_f as follows.

(i) $P_f(G) = \{v \in V(G): f(v) = +1\}$

(ii) $M_f(G) = \{v \in V(G): f(v) = -1\}$

(iii) $N_f(G) = \left\{v \in V(G): \sum_{u \notin N[v]} f(u) \geq 1\right\}$

ISSN 2229-5518

REMARK 2.4

If f is any complementary majority dominating function of a graph G of order n, then it is obvious that $|P_f| + |M_f| = n$ and $\gamma_{Cmaj}(G) = |P_f| - |M_f|$.

THEOREM 2.5

For stars of order $n \geq 2$, we have

$$\gamma_{cmaj}(K_{1,n}) = \begin{cases} -1 & \text{if n is even,} \\ 2 & \text{if n is odd.} \end{cases}$$

Proof.

Case 1. n is even.

Let u be the central vertex of $K_{1,n}$ and let $v_1, v_2, \ldots, v_n$ be the other vertices. Define $f: V \longrightarrow \{-1, +1\}$ by $f(u) = -1$ and

$$f(v_i) = \begin{cases} -1 & \text{if } 1 \leq i \leq \frac{n}{2} \\ +1 & \text{otherwise} \end{cases}$$

Claim: f is a complementary majority dominating function.

Let v be a pendant vertex with $f(v) = -1$. Then

$$\sum_{u \notin N[v]} f(u) = (-1)\left(\frac{n}{2} - 1\right) + \frac{n}{2} = 1$$

Since there are $\frac{n}{2}$ vertices with $f(v_i) = -1$, f is a complementary majority dominating function with

$$\sum_{v \in V(K_{1,n})} f(v) = -1 + (-1)\left(\frac{n}{2}\right) + \frac{n}{2} = -1$$

If the number of vertices with function value -1 is increased by 1, then all the pendant vertices will not satisfy the condition necessary for a complementary majority dominating function. Therefore -1 is the minimum value of $\sum_{v \in V(K_{1,n})} f(v)$. Hence $\gamma_{Cmaj}(K_{1,n}) = -1$ if n is even.

Case 2. n is odd.

Define $f: V \longrightarrow \{-1, +1\}$ by $f(u) = -1$ and

$$f(v_i) = \begin{cases} -1 & \text{if } 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor - 1 \\ +1 & \text{otherwise} \end{cases}$$

Claim: f is a complementary majority dominating function.

Let v be a pendant vertex with $f(v) = +1$. Then

$$\sum_{u \notin N[v]} f(u) = (-1)\left(\left\lfloor \frac{n}{2} \right\rfloor - 1\right) + \left\lceil \frac{n}{2} \right\rceil = 2$$

Since there are $\left\lceil \frac{n}{2} \right\rceil$ vertices with $f(v_i) = +1$, f is a complementary majority dominating function with

$$\sum_{v \in V(K_{1,n})} f(v) = -1 + (-1)\left(\left\lfloor \frac{n}{2} \right\rfloor - 1\right) + \left\lceil \frac{n}{2} \right\rceil + 1 = 2$$

If the number of vertices with function value -1 is increased by 1, then only the pendant vertices with $f(v_i) = -1$ satisfy the condition necessary for a complementary majority dominating function. Therefore 2 is the minimum value of $\sum_{v \in V(K_{1,n})} f(v)$. Hence $\gamma_{cmaj}(K_{1,n}) = 2$ if n is odd.

THEOREM 2.6

For $n \geq 6$, we have

$$\gamma_{cmaj}(P_n) = \begin{cases} 0 & \text{if n is even,} \\ 1 & \text{if n is odd.} \end{cases}$$

Proof. Let $v_1, v_2, \ldots, v_n$ be the vertices of P_n.

Case 1. n is even.

Define $f: V \longrightarrow \{-1, +1\}$ by

$$f(v_i) = \begin{cases} -1 & \text{if } 1 \leq i \leq \frac{n}{2} \\ +1 & \text{otherwise} \end{cases}$$

We claim that f is a complementary majority dominating function.

For $i = 1$

$$\sum_{u \notin N[v_1]} f(u) = (-1)\left(\frac{n-4}{2}\right) + 1\left(\frac{n}{2}\right) = 2$$

For $2 \leq i \leq \frac{n}{2} - 1$

$$\sum_{u \notin N[v_i]} f(u) = (-1)\left(\frac{n-6}{2}\right) + 1\left(\frac{n}{2}\right) = 3$$

For $i = \frac{n}{2}$

$$\sum_{u \notin N[v_{\frac{n}{2}}]} f(u) = (-1)\left(\frac{n}{2} - 2\right) + 1\left(\frac{n}{2} - 1\right) = 1$$

Therefore f is a complementary majority dominating function. Since $\sum_{u \notin N[v_{\frac{n}{2}}]} f(u) = 1$, the labelling is minimum with respect to the vertices $1 \leq i \leq \frac{n}{2} - 2$ and $\frac{n}{2} + 2 \leq i \leq n$. If $f(v_{\frac{n}{2}+1}) = -1$, then $\sum_{u \notin N[v_1]} f(u) < 1$. It is easy to observe that

$$\sum_{v \in V(P_n)} f(v) = (-1)\left(\frac{n}{2}\right) + 1\left(\frac{n}{2}\right) = 0$$

ISSN 2229-5518

Hence $\gamma_{Cmaj}(P_n) = 0$ if n is even.

Case 2. n is odd

Define $f: V \longrightarrow \{-1, +1\}$ by

$$f(v_i) = \begin{cases} -1 & \text{if } i = 1 \text{ and } 3 \leq i \leq \left\lceil \frac{n}{2} \right\rceil \\ +1 & \text{otherwise} \end{cases}$$

We claim that f is a complementary majority dominating function.

For $i = 1$

$$\sum_{u \notin N[v_1]} f(u) = (-1)\left(\frac{n-3}{2}\right) + 1\left(\left\lfloor \frac{n}{2} \right\rfloor\right) = 1$$

For $i = 2,3, \left\lceil \frac{n}{2} \right\rceil$

$$\sum_{u \notin N[v_i]} f(u) = (-1)\left(\frac{n-3}{2} - 1\right) + 1\left(\left\lfloor \frac{n}{2} \right\rfloor\right) = 2$$

For $4 \leq i \leq \left\lceil \frac{n}{2} \right\rceil - 1$

$$\sum_{u \notin N[v_i]} f(u) = (-1)\left(n - \left\lfloor \frac{n}{2} \right\rfloor - 4\right) + 1\left(\left\lfloor \frac{n}{2} \right\rfloor + 1\right) = 4$$

Therefore f is a complementary majority dominating function. Since $\sum_{u \notin N[v_1]} f(u) = 1$, the labelling is minimum with respect to the vertices v_i, $3 \leq i \leq n$. If $f(v_2) = -1$, then $\sum_{u \notin N[v_{\lceil \frac{n}{2} \rceil}]} f(u) < 1$. It is easy to observe that

$$\sum_{v \in V(P_n)} f(v) = (-1)\left(\left\lceil \frac{n}{2} \right\rceil - 1\right) + 1\left(\left\lfloor \frac{n}{2} \right\rfloor + 1\right) = 1$$

Hence $\gamma_{Cmaj}(P_n) = 1$ if n is odd.

THEOREM 2.7

For $n \geq 4$, we have

$$\gamma_{cmaj}(C_n) = \begin{cases} 0 & \text{if n is even,} \\ 3 & \text{if n is odd.} \end{cases}$$

Proof. Let $v_1, v_2, \ldots, v_n$ be the vertices of C_n.

Case 1. n is even.

Define $f: V \longrightarrow \{-1, +1\}$ by

$$f(v_i) = \begin{cases} -1 & \text{if } 1 \leq i \leq \frac{n}{2} \\ +1 & \text{otherwise} \end{cases}$$

We claim that f is a complementary majority dominating function.

For $i = 1$ and $i = \frac{n}{2}$

$$\sum_{u \notin N[v_i]} f(u) = (-1)\left(\frac{n}{2} - 2\right) + 1\left(\frac{n}{2} - 1\right) = 1$$

For $2 \leq i \leq \frac{n}{2} - 1$

$$\sum_{u \notin N[v_i]} f(u) = (-1)\left(\frac{n-6}{2}\right) + 1\left(\frac{n}{2}\right) = 3$$

Therefore f is a complementary majority dominating function.

Since $\sum_{u \notin N[v_i]} f(u) = 1$, for i=1, $\frac{n}{2}$ the labelling is minimum with respect to all the vertices v_i.

It is easy to observe that

$$\sum_{v \in V(C_n)} f(v) = (-1)\left(\frac{n}{2}\right) + 1\left(\frac{n}{2}\right) = 0$$

Hence $\gamma_{Cmaj}(C_n) = 0$ if n is even.

Case 2. n is odd

Define $f: V \longrightarrow \{-1, +1\}$ by

$$f(v_i) = \begin{cases} -1 & \text{if } i = 1, n \text{ and } 3 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor \\ +1 & \text{otherwise} \end{cases}$$

We claim that f is a complementary majority dominating function.

For $i = 1,2,3, \left\lfloor \frac{n}{2} \right\rfloor, n$

$$\sum_{u \notin N[v_i]} f(u) = (-1)\left(n - 3 - \left\lceil \frac{n}{2} \right\rceil\right) + 1\left(\left\lfloor \frac{n}{2} \right\rfloor\right) = 2$$

For $4 \leq i \leq \left\lceil \frac{n}{2} \right\rceil - 1$

$$\sum_{u \notin N[v_i]} f(u) = (-1)\left(n - 3 - \left\lceil \frac{n}{2} \right\rceil\right) + 1\left(\left\lceil \frac{n}{2} \right\rceil\right) = 4$$

Therefore f is a complementary majority dominating function.

Since $\sum_{u \notin N[v_i]} f(u) = 2$, for i=1, $\left\lfloor \frac{n}{2} \right\rfloor$ and $\deg v_i = 2$, for all i, the labelling is minimum with respect to all the vertices v_i. It is easy to observe that

$$\sum_{v \in V(C_n)} f(v) = (-1)\left(\left\lceil \frac{n}{2} \right\rceil - 1\right) + 1\left(\left\lfloor \frac{n}{2} \right\rfloor + 1\right) = 1$$

Hence $\gamma_{Cmaj}(C_n) = 1$ if n is odd.

ISSN 2229-5518

Next, we consider the complete multipartite graph with m parts, each of order 2.

THEOREM 2.8

$$\gamma_{cmaj}\left(K_2^{(m)}\right) = 0$$

Proof. Let G be a graph which is isomorphic to $K_2^{(m)}$

Let V be a vertex of G. Suppose g is a complementary majority dominating function of G such that $\gamma_{cmaj}(G)$=g(V). Then $\sum_{u \notin N[v_i]} f(u)=1$ for m vertices in G. Since G is isomorphic to $K_2^{(m)}$, we have $|P_g|$=m and $|M_g|$=m. Hence $\gamma_{cmaj}(G) \geq 0$.

On the other hand , define a function $f: V \longrightarrow \{-1,+1\}$ by

Case 1. m is odd.

$$f(v_i) = \begin{cases} -1 & \text{for } \left\lfloor \frac{m}{2} \right\rfloor \text{ partite classes of V and} \\ & \text{one vertex in } \left\lfloor \frac{m}{2} \right\rfloor + 1 \text{ partite class} \end{cases}$$

and assign +1 for remaining vertices.

Then f is complementary majority dominating function of G. Hence

$$\gamma_{cmaj}(G) \leq -1\left(\left\lfloor \frac{m}{2} \right\rfloor\right) + (-1)(1) + (`1)\left(\left\lfloor \frac{m}{2} \right\rfloor\right) + 1(1) = 0.$$

Case 2. m is even.

$$f(v_i) = \begin{cases} 1 & \text{for } \frac{m}{2} \text{ partite classes of V} \\ -1 & \text{otherwise} \end{cases}$$

Then f is complementary majority dominating function of G. Hence $\gamma_{cmaj}(G) \leq -1\left(\frac{m}{2}\right) + 1\left(\frac{m}{2}\right) = 0$.

Consequently the result follows.

THEOREM 2.9

Let G denote the friendship graph with t triangles. The

$$\gamma_{cmaj}(G) = \begin{cases} -1 & \text{if t is even,} \\ 1 & \text{if t is odd.} \end{cases}$$

Proof. Let u be the central vertex of G and

let $(v_1, v_2, \ldots, v_{2t})$ be the vertices in the triangles.

Case 1. t is odd

Define $f: V \longrightarrow \{-1,+1\}$ by f(u)=-1 and

$$f(v_i) = \begin{cases} +1 & \text{if } 1 \leq i \leq t \text{ and } i = t+2 \\ -1 & \text{otherwise} \end{cases}$$

Then it is easy to verify that $\sum_{u \notin N[v_i]} f(u) > 1$ for $t \leq i \leq 2t$.

Hence f is a complementary majority dominating function of G. Therefore

$$\gamma_{cmaj}(G) \leq -1(t) + 1(t+1) = 1.$$

Now suppose g be a complementary majority dominating function of G such that $\gamma_{cmaj}(G) = g(V)$. Then there exists a vertex $v_i \in N_g$ such that $\sum_{u \notin N[v_i]} g(u) = 2$.

If the vertex v_i and the vertex adjacent with v_i namely v_j are assigned with -1, then only t-1 vertices belongs to N_g, which is a contradiction. Now suppose $g(v_i)$=-1 and $g(v_j)$=+1, then t+1 vertices belongs to N_g. Hence $\gamma_{cmaj}(G) \geq \sum_{u \notin N[v_i]} g(u) + g(v_i) + g(v_j) + (-1) = 1$.

Case 2. t is even

Define $f: V \longrightarrow \{-1,+1\}$ by f(u)= -1 and

$$f(v_i) = \begin{cases} +1 & \text{if } 1 \leq i \leq t \\ -1 & \text{otherwise} \end{cases}$$

Then it is easy to verify that $\sum_{u \notin N[v_i]} f(u) = 2$ for $t+1 \leq i \leq 2t$. Hence f is a complementary majority dominating function of G. Therefore

$$\gamma_{cmaj}(G) \leq -1(t) + 1(t) + (-1)(1) = -1.$$

Now suppose g be a complementary majority dominating function of G such that $\gamma_{cmaj}(G) = g(V)$. Then there exists a vertex $v_i \in N_g$ such that $\sum_{u \notin N[v_i]} g(u) = 2$.

Hence $\gamma_{cmaj}(G) \geq \sum_{u \notin N[v_i]} g(u) - 2 - 1 = -1$. Consequently the result follows.

THEOREM 2.10

Let $G = K_{2n} - M$; where M is a perfect matching in the complete graph K_{2n}. Then $\gamma_{Cmaj}(G) = 0$.

Proof. Let $V(K_{2n}) = (v_1, v_2, \ldots, v_{2n})$ and

$$M = (v_1v_2, v_3v_4, v_5v_6, \ldots, v_{2n-1}v_{2n}).$$

Now define a function $f: V \to \{-1,1\}$ by

$$f(v_i) = \begin{cases} +1 & \text{if i is even} \\ -1 & \text{otherwise} \end{cases}$$

We claim that f is a complementary majority dominating function.

For i = 1

$$\sum_{u \notin N[v_1]} f(u) = f(v_2) = 1$$

For i = 3

$$\sum_{u \notin N[v_3]} f(u) = f(v_4) = 1$$

Similarly For $i = 5, 7, \ldots, 2n-1$

$$\sum_{u \notin N[v_i]} f(u) = f(v_{i+1}) = 1$$

Therefore f is a complementary majority dominating function.

Since $\sum_{u \notin N[v_i]} f(u) = 1$, for i is odd, the labelling is minimum with respect to all the vertices v_i. Also if any of the vertex v_i, (i is even) is assigned with -1, then $\sum_{u \notin N[v_i]} f(u) = f(v_j) = -1$, for some j is odd. Therefore f is minimum. It is easy to observe that

$$\sum_{v \in V(G)} f(v) = (-1)(n) + 1(n) = 0$$

Hence $\gamma_{Cmaj}(G) = 0$.

3 Open problems

(i) Given any integer k, does there exist graph G such that $\gamma_{CS}(G) - \gamma_{Cmaj}(G) = k$.

(ii) Characterization of graphs G for which $\gamma_{Cmaj}(G) = \gamma_{CS}(G)$.

(iii) Find both lower and upper bound for γ_{Cmaj}.

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